MATH 320 Unit 5 Exercises

Morphisms, Ideals, and Congruence

Let R be a ring and $S \subseteq R$. We call S a *subring* of R if it is closed under addition and multiplication, contains 0_R, and for every $a \in S$ the solution of $a + x = 0_R$ is in S (not just in R).

Let R, S be rings, and $f : R \to S$ a function. We call f a *homomorphism* if it satisfies

$$\text{For all } a, b \in R,\ f(a+b) = f(a) + f(b) \text{ and } f(ab) = f(a)f(b).$$

If a homomorphism is also a bijection (i.e. is surjective and injective), we call it a *isomorphism*, and say that the rings are *isomorphic*.

Basic Homomorphism Properties Theorem: Let $f : R \to S$ be a homomorphism. Then (i) $f(0_R) = 0_S$; and (ii) For all $a \in R$, $f(-a) = -f(a)$; and (iii) For all $a, b \in R$, $f(a-b) = f(a) - f(b)$.

Surjective Homomorphism Properties Theorem: Let $f : R \to S$ be a surjective homomorphism, and suppose 1_R exists. Then (i) $f(1_R) = 1_S$; and (ii) If $u \in R$ is a unit (in R), then $f(u)$ is a unit in S and $f(u)^{-1} = f(u^{-1})$.

Ring Image Theorem: Let $f : R \to S$ be a homomorphism. Then $Im(f) = \{f(r) : r \in R\}$ is a subring of S.

Let I be a subring of R. We call I an *ideal* if it also satisfies

$$\text{For all } r \in R, a \in I, \text{ then } ra \in I \text{ and } ar \in I.$$

Let R be a commutative ring with identity, let $c \in R$. We define $\{rc : c \in R\}$, writing (c). We call this subset of R the *principal ideal generated by c*.

Let R be a ring with ideal I, and let $a, b \in R$. We say that *a is congruent to b modulo I*, writing $a \equiv b \pmod{I}$, if $a - b \in I$.

Let R be a ring with ideal I, and let $a \in R$. The *congruence class (or equivalence class, or coset) of a modulo I*, written $a + I$, is the set $\{b \in R : b \equiv a \pmod{I}\}$. We define R/I to be the set of equivalence classes modulo I.

<u>For Nov. 6:</u>

1. Prove that $f : \mathbb{C} \to \mathbb{C}$ given by $f(a + bi) = a - bi$ is an isomorphism.

2. Let R be a ring and let $R^\star = \{(a, a) : a \in R\}$, a subring of $R \times R$. Prove that $f : R \to R^\star$ given by $f(a) = (a, a)$ is an isomorphism.

3. Let R, S be rings and let $R^\dagger = \{(a, 0_S) : a \in R\}$, a subring of $R \times S$. Prove that $f : R \to R^\dagger$ given by $f(a) = (a, 0_S)$ is an isomorphism.

4. Recall the ring from Unit 2 given by $R = \left\{ \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right) : a, b \in \mathbb{R} \right\}$. Prove that $f : R \to \mathbb{C}$ given by $f \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right) = a + bi$ is an isomorphism.

<u>For Nov. 13:</u>

5. Let R, S be nontrivial rings. Prove that the zero map $z : R \to S$ given by $z(r) = 0_S$ (for all r) is a homomorphism, that is neither surjective nor injective.

6. Let $n \in \mathbb{Z}$ with $n \geq 2$. Prove that the function $f : \mathbb{Z} \to \mathbb{Z}_n$ given by $f(a) = [a]$ is a homomorphism, that is surjective but not injective.

7. Recall the ring from Unit 2 given by $M_2(\mathbb{R}) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) : a, b, c, d \in \mathbb{R}\}$. Prove that the function $f : \mathbb{R} \to M_2(\mathbb{R})$ given by $f(r) = (\begin{smallmatrix} 0 & 0 \\ -r & r \end{smallmatrix})$ is a homomorphism, that is injective but not surjective.

8. Prove the Basic Homomorphism Properties Theorem.

<u>For Nov. 18:</u>

9. In the ring $\mathbb{Z}[x]$, let I be those polynomials whose constant terms are even. Prove that I is an ideal in $\mathbb{Z}[x]$.

10. Prove that the ideal from Problem 9 is not principal.

11. Let R be a ring, and let I, J be ideals. Prove that $I \cap J$ is an ideal of R.
    For $R = \mathbb{Z}$, $I = (m)$, $J = (n)$, it turns out that $I \cap J = (k)$ for some $k \in \mathbb{Z}$. Determine k.

12. Let R be a ring, and let I, J be ideals. Define $I + J = \{a + b : a \in I, b \in J\}$. Prove that $I + J$ is an ideal of R, and also that $I \subseteq I + J$ (also $J \subseteq I + J$, but no need to prove this).
    For $R = \mathbb{Z}$, $I = (m)$, $J = (n)$, it turns out that $I + J = (k)$ for some $k \in \mathbb{Z}$. Determine k.

<u>For Nov. 20:</u>

13. Let R be a ring, and I an ideal. Let $a, b, c \in R$, and suppose that $a \equiv b \pmod{I}$. Prove that $a + c \equiv b + c \pmod{I}$ and also $ac \equiv bc \pmod{I}$.

14. Let R be a ring, and I an ideal. Prove that equivalence modulo I is reflexive, symmetric, and transitive. That is, prove that $\forall a, b, c \in R$, (i) $a \equiv a$; and (ii) if $a \equiv b$ then $b \equiv a$; and (iii) if $a \equiv b$ and $b \equiv c$ then $a \equiv c$.

15. Verify that $I = \{0, 3\}$ is an ideal in $\mathbb{Z}_6$ and list all its cosets.

16. Let $R = \mathbb{Z}$ and $I = (3)$, the principal ideal generated by 3. Prove that $R/I = \{0 + I, 1 + I, 2 + I\}$, and that it is isomorphic to $\mathbb{Z}_3$.

Extra:

17. Prove that $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic.

18. Prove that $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ are isomorphic.
HINT: Start with $f(0) = (0, 0)$ and $f(1) = (1, 1)$ and extend to a function.

19. Prove the Surjective Homomorphism Properties Theorem.

20. Prove the Ring Image Theorem.

21. Let R be a ring. Prove that $I_1 = \{0_R\}$ and $I_2 = R$ are both ideals of R.

22. Let R be a ring, and let $I \subseteq R$ be nonempty. Prove that I is an ideal if and only if it satisfies (i) $\forall a, b \in I,\ a - b \in I$; and (ii) $\forall r \in R, \forall a \in I,\ ra \in I$ and $ar \in I$.

23. Let R be an integral domain, and $a, b \in R$. Prove that $(a) = (b)$ if and only if $a = bu$ for some unit $u \in R$.

24. Let R be a commutative ring with identity, and let $c_1, c_2, \ldots, c_n \in R$. Set $(c_1, c_2, \ldots, c_n) = \{r_1c_1 + r_2c_2 + \cdots + r_nc_n | r_1, r_1, \ldots, r_n \in R\}$. Prove that $(c_1, c_2, \ldots, c_n)$ is an ideal in R.
Note: in particular, taking $n = 1$, this proves that a principal ideal is an ideal.

25. Let $R = \mathbb{Z}[x]$, and let $I = (2, x)$. Prove that I is the same as in Problem 9, and prove that $R/I = \{0 + I, 1 + I\}$.